(Di)graphs products, labelings and related results

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Abstract

Gallian's survey shows that there is a big variety of labelings of graphs. By means of (di)graphs products we can establish strong relations among some of them. Moreover, due to the freedom of one of the factors, we can also obtain enumerative results that provide lower bounds on the number of nonisomorphic labelings of a particular type. In this talk, we will focus in three of the (di)graphs products that have been used in these duties: the $\otimes_h$-product of digraphs, the weak tensor product of graphs and the weak $\otimes_h$-product of graphs.

Keywords: $\otimes_h$-product, weak tensor product of graphs, weak $\otimes_h$-product, (super) edge-magic, α-labeling

1 Introduction

For the undefined notation and terminology, we refer the reader to either [7,27] or [11]. We say that G is a (p,q)-graph when $|V(G)| = p$ and $|E(G)| = q$ and

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we let $[1,n]=\{1,2,\ldots,n\}$. Let G be a (p,q)-graph. A *β-labeling* of G is an injective function $f: V(G)\rightarrow [0,q]$ such that the induced edge labeling $g: E(G)\rightarrow [1,q]$ defined by $g(uv)=|f(u)-f(v)|$ is also an injective function. This type of labeling, also known as *graceful* labeling [12], was introduced by Rosa [23] in the context of graph decompositions. A β-labeling f of G is said to be an *α-labeling* if there exists a constant k, called the *characteristic* of f, such that $\min\{f(u),f(v)\}\leq k<\max\{f(u),f(v)\}$, for every edge $uv\in G$. If $f: V(G)\rightarrow [0,q]$ is an α-labeling of G, then $L=\{u: \ f(u)\leq k\}$ and $H=\{u: \ f(u)>k\}$ defines a partition of $V(G)$ into two stable sets. El-Zanati et al. introduced in [8] a relaxation of an α-labeling. A graceful labeling f of G is a *near α-labeling* if there exists a partition $V(G)=A\cup B$ with the property that each edge of G is of the form uv with $u\in A$ and $v\in B$ and $f(u)<f(v)$. In 1998, Enomoto, Lladó, Nakamigawa and Ringel [9] introduced the concept of super edge-magic labelings and super edge-magic graphs. Previously, in 1991 Acharya and Hegde introduced the concept of *strongly indexable* graphs in [1]. It turns out that the sets of super edge-magic graphs and of strongly indexable graphs are the same. Let G be a (p,q)-graph and let $f: V(G)\cup E(G)\rightarrow [1,p+q]$ be a bijection that meets the following conditions: (i) $f(V(G))=[1,p]$ and (ii) $f(u)+f(uv)+f(v)=k$, for all $uv\in E(G)$. Then f is called a *super edge-magic labeling* of G and G is called a *super edge-magic graph*. Super edge-magic labelings are a special case of *edge-magic* labelings defined in [17] by Kotzig and Rosa. For further information on labelings of the magic (and the antimagic) type, the reader is referred to [2,26].

In [10], Figueroa-Centeno, Ichishima, Muntaner-Batle and Rius-Font introduced the following product of digraphs. Let D be a digraph and let $\Gamma=\{F_i\}_{i=1}^m$ be a family of digraphs such that $V(F_i)=V$, for every $i\in[1,m]$. Consider any function $h: E(D)\longrightarrow \Gamma$. Then the product $D\otimes_h\Gamma$ is the digraph with vertex set $V(D)\times V$ and $((a,b),(c,d))\in E(D\otimes_h\Gamma)$ if and only if $(a,c)\in E(D)$ and $(b,d)\in E(h(a,c))$. The adjacency matrix of $D\otimes_h\Gamma$ is obtained by multiplying every 0 entry of $A(D)$, the adjacency matrix of D, by the $|V|\times|V|$ null matrix and every 1 entry of $A(D)$ by $A(h(a,c))$, where (a,c) is the arc related to the corresponding 1 entry. Notice that when h is constant, the adjacency matrix of $D\otimes_h\Gamma$ is just the classical Kronecker product $A(D)\otimes A(h(a,c))$. When $|\Gamma|=1$, we just write $D\otimes\Gamma$. Given two bipartite graphs G and F with stable sets L_G, H_G, L_F and H_F, respectively, Snevily [25] defines the *weak tensor product* $G\bar{\otimes}F$ as the bipartite graph with vertex set $(L_G\times L_F, H_G\times H_F)$ and with $(a,x)(b,y)$ being an edge if and only if $ab\in E(G)$ and $xy\in E(F)$. Thus, it comes from the tensor product (also known as direct product) of two graphs by deleting some of its vertices and edges, according to the stable sets of the two graphs involved. Inspirated by the definition of the $\otimes_h$-product, the weak $\otimes_h$-product of graphs was introduced in [21]. Let G be a bipartite graph with stable sets L_G and H_G and let Γ be a family of bipartite graphs such that $V(F) = L \cup H$, for every $F \in \Gamma$. Consider any function $h : E(G) \to \Gamma$. Then, the product $G\bar{\otimes}_h\Gamma$ is the graph with vertex set $(L_G \times L, H_G \times H)$ and $(a, x)(b, y) \in E(G\bar{\otimes}_h\Gamma)$ if and only if $ab \in E(G)$ and $xy \in E(h(ab))$.

In this talk, we will focus on the (di)graphs products mentioned above when they are used either for constructing new families of labeled (di)graphs, or to obtain strong relation among labelings, or enumerative results that provide lower bounds for the number of nonisomorphic labelings of a particular type. We will also show an application to construct Langford sequences.

2 The $\otimes_h$-product applied to labelings

The first paper that uses the $\otimes_h$-product for constructing labelings is [10]. As in [10], a digraph D is said to admit a labeling l if its underlying graph, und(D), admits l. Almost all results contained in [10] use as the second factor of the product the set of SEM 1-regular digraphs of odd order n, that is denoted by $\mathcal{S}_n$. It turns out that many of the results in [10] also hold when instead of considering the set of SEM 1-regular digraphs, we consider families of SEM labeled digraphs with size equal to order, provided that the magic sum for each element of the family is constant. A super edge-magic labeled digraph F is in the set $\mathcal{S}_n^k$ if $|V(F)| = |E(F)| = n$ and the minimum sum of the labels of the adjacent vertices is equal to k. Since the minimum sum of the labels of adjacent vertices in a super edge-magic labeled cycle equals to $(n + 3)/2$, we have $\mathcal{S}_n \subset \mathcal{S}_n^{(n+3)/2}$ and, therefore, the family $\mathcal{S}_n^k$ is a generalization of the family $\mathcal{S}_n$.

Theorem 2.1 *[20] Assume that D is any (super) edge-magic digraph and h is any function $h : E(D) \to \mathcal{S}_n^k$. Then und$(D \otimes_h \mathcal{S}_n^k)$ is (super) edge-magic.*

Analogous results can be found in [20] when instead of assuming D (super) edge-magic we assume that D is one of the following types of labelings: (super) edge bi-magic [3,4], harmonious [14], sequential [13], partitional [16], cordial [6]. Almost all correspond to generalizations of previous results found in [15,19].

Let $M = (a_{i,j})$ be a square matrix of order n. The matrix $(a^R_{i,j})$ is the *rotation of the matrix M*, denoted by M^R, when $a^R_{i,j} = a_{n+1-j,i}$. Graphically this corresponds to a rotation of the matrix by $\pi/2$ radiants clockwise. A digraph S is said to be a *rotation super edge-magic digraph of order n and minimum sum k*, if its adjacency matrix is the rotation of the adjacency matrix of an element in $\mathcal{S}_n^k$. We denote by $\mathcal{RS}_n^k$ the set of all digraphs that are rotation super edge-magic digraphs of order n and minimum sum k.

Bloom and Ruiz introduced in [5] a generalization of graceful labelings, that they called k-equitable labelings. The next result is an application of the $\otimes_h$-product to k-equitable digraphs.

Theorem 2.2 *[20] Let D be an (optimal) k-equitable digraph and let $h : E(D) \to \mathcal{RS}_n^{(n+3)/2}$ be any function. Then $D \otimes_h \mathcal{RS}_n^{(n+3)/2}$ is (optimal) k-equitable.*

2.1 Related results

Let d be a positive integer. A *Langford sequence* of order m and defect d [24] is a sequence $(l_1, l_2, \ldots, l_{2m})$ of $2m$ numbers such that (i) for every $k \in [d, d+m-1]$ there exist exactly two subscripts $i, j \in [1, 2m]$ with $l_i = l_j = k$, (ii) the subscripts i and j satisfy the condition $|i-j| = k$. Langford sequences, for $d = 2$, were introduced in [18] and they are referred as *perfect Langford sequences*. Theorem 2.2 was used in [22] to construct an exponential number of Langford sequences with certain order and defect.

3 The weak tensor and the weak $\otimes_h$-product applied to labelings

Snevily proves the next result.

Theorem 3.1 *[25] Let G and F be two bipartite graphs that have α-labelings, with stable sets L_G, H_G, L_F and H_F, respectively. Then, the graph $G\bar{\otimes}F$ also has an α-labeling.*

Using a similar proof, Theorem 3.1 was extended to near α-labelings in [8]. The next result generalizes Theorem 3.1 by introducing the $\bar{\otimes}_h$-product of graphs.

Theorem 3.2 *[21] Let G be a bipartite graph that has an α-labeling. Let Γ be a family of bipartite graphs such that for every $F \in \Gamma$, $|E(F)| = n$ and there exists and α-labeling f_F with $f_F(V(F)) = L \cup H$, where $L, H \subset [0, n]$ are the stable sets defined by the characteristic of f_F and they do not depend* *on F. Consider any function $h : E(G) \to \Gamma$. Then, the graph $G\bar{\otimes}_h\Gamma$ also has an α-labeling.*

In turns out that a similar result to Theorem 3.2 also holds when instead of considering graphs with α-labelings, we consider graphs that admit either a near α-labeling or a bigraceful labeling.

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